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Superdiffusivity of Occupation-Time Variance in 2-dimensional Asymmetric Exclusion Processes with Density $\rho = 1/2$*

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We compute that the growth of the occupation-time variance at the origin up to time t in dimension $d = 2$ with respect to asymmetric simple exclusion in equilibrium with density $\rho = 1/2$ is in a certain sense at least $t \log(\log t)$ for general rates, and at least $t(\log t)^{1/2}$ for rates which are asymmetric only in the direction of one of the axes. These estimates give a complement to bounds in the literature when $d = 1$, and are consistent with an important conjecture with respect to the transition function and variance of "second-class" particles.

KEY WORDS: Variance of occupation times, Second-class particle, Simple exclusion
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1. INTRODUCTION AND RESULTS

Consider asymmetric exclusion processes on $\mathbb{Z}^d$ whose rates have a drift. It is known that the occupation-time variance at the origin up to time t in equilibrium is proportional to t times the expected time a second-class particle, beginning at the origin, spends at the origin, that is, $t \int_0^t (1 - s/t) p_s(0, 0) ds$ where $p_s(0, j)$ is the second-class particle transition function (1.3). Let us now fix the equilibrium density $\rho = 1/2$ so that the mean of the second-class particle at time t, proportional to $(1 - 2\rho)t$, vanishes. Recently, it has been argued, as the variance of a second-class particle at time t, starting initially at the origin– $\sum j^2 p_t(0, j)$ in this case– is conjectured to be on the order $t^{4/3}$ in $d = 1^{(13)}$ and proved (for a closely related resolvent quantity) to be at least $t^{5/4\,(4)}$, that the transition function of the second-class particle decays on order $t^{-2/3}$ in $d = 1$ (cf. Eq. (4.8) Ref. (7)). In

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$d = 2$, the second-class particle variance is conjectured as $O(t(\log t)^{2/3})$[13] with a proof (for a resolvent quantity when the process rates are asymmetric only in the direction of one of the axes)[14]. Perhaps by the same sort of reasoning as in Ref. (7), one may claim the second-class transition function decays as $t^{-1}(\log t)^{-1/3}$ in $d = 2$ (cf. Eq. (12) Ref. (13)). Then, the occupation-time variance orders should match second-class particle variance orders in both $d = 1$ and 2. We mention also these variance orders have connections to fluctuation orders of the current across the origin and in turn to certain Riemann-Hilbert and combinatorial problems on which there has been much recent study (cf. Refs. (2, 7, 14)).

In fact, in Ref. (1), it was shown recently in $d = 1$ that the occupation-time variance at the origin diverges in a sense when density $\rho = 1/2$ at least on order $t^{5/4}$ which is consistent with the above discussion. The purpose of this note is to give analogous consistent bounds in $d = 2$, namely, we show the occupation-time variance at density $\rho = 1/2$ diverges in $d = 2$ on order at least as $t \log(\log t)$ for general asymmetric rates, and at least as $t(\log t)^{1/2}$ when the asymmetry is only in the direction of one of the axes (Proposition 1.3). The methods are to link occupation-time variances and certain resolvent H_{-1} norms, and then to use some "free-particle" comparisons of H.T. Yau in the style of Bernardin[1]. These techniques seem well suited to our case $\rho = 1/2$. But, we remark, open and of interest, would be to study the more general problem of occupation time in a moving frame with velocity given by the mean drift of the second-class particle (cf. after (1.3)) when density $\rho \neq 1/2$.

1.1. Model

Informally, the simple exclusion process on $\mathbb{Z}^d$ is a collection of random walks which move with jump rates $p(i, i + j) = p(j)$ independently except in that jumps to occupied vertices are suppressed. In this article, to avoid technicalities, we will assume p is finite-range, that is, for some $R < \infty$, $p(i) = 0$ for $|i| > R$, and also that its symmetrization $(p(\cdot) + p(-\cdot))/2$ is irreducible. More formally, let $\Sigma = \{0, 1\}^{\mathbb{Z}^d}$ be the configuration space where a configuration $\eta = \{\eta_i : i \in \mathbb{Z}^d\}$ is a collection of "occupation" coordinates where $\eta_i = 1$ if i is occupied and $\eta_i = 0$ otherwise. The exclusion process is a Markov process $\eta(t)$ evolving on Σ with formal generator

$$(Lf)(\eta) = \sum_j \sum_i p(j)\eta_i(1 - \eta_{i+j})(f(\eta^{i,i+j}) - f(\eta)).$$

Here, $\eta^{i,i+j}$ is the configuration obtained from η by interchanging the values at i and $i + j$. Let also T_t denote the associated semi-group. See Ref. (5) for more details.

It is well-known that there is a family of invariant measures $\{P_\rho : 0 \leq \rho \leq 1\}$ each of which concentrate on configurations of a fixed density ρ. These measures take form as Bernoulli product measures, that is, P_ρ independently places a particle at each vertex with probability ρ. Let E_ρ denote the process expectation with respect to P_ρ. Denote also by $\langle\cdot,\cdot\rangle_\rho$ and $\|\cdot\|_0$ the innerproduct and norm on $L^2(P_\rho)$.

In the following, we will work with the $L^2(P_\rho)$ extension of T_t and its generator L_ρ which is the closure of L over local functions, that is functions depending only on a finite number of coordinates. One can compute that the adjoint L^*_ρ, with respect to P_ρ, is itself the generator of simple exclusion but with reversed jump rates $p(-\cdot)$. See Proposition IV.4.1, Ref. (5)) for more discussion. For simplicity, we will drop the suffixes in this notation, $L = L_\rho$ and $L^* = L^*_\rho$.

1.2. General Problem and Connection to Second-Class Particles

Consider the centered occupation time, say, at the origin up to time t, $A_\rho(t) = \int_0^t(\eta_0(s)-\rho)ds$. The problem is to compute the variance of $A_\rho(t)$ under the equilibrium P_ρ. Let $\sigma_t^2 = E_\rho[A_\rho^2(t)]$ denote the variance. We compute, using stationarity and basic calculations, that

$$\begin{aligned}\sigma_t^2 &= 2\int_0^t\int_0^u E_\rho[(\eta_0(s)-\rho)(\eta_0(u)-\rho)]\,dsdu\\ &= 2\int_0^t (t-s)E_\rho[(\eta_0(s)-\rho)(\eta_0(0)-\rho)]\,ds. \qquad (1.1)\end{aligned}$$

To express the kernel further, consider the "basic coupling" of two systems, the first starting under $\xi \sim P_\rho(\cdot|\eta_0 = 0)$ and the second under $\xi + \delta_0$, that is with an extra particle at the origin (where δ_0 is the configuration with exactly one particle at the origin). Let $(\xi(t), R(t)) \sim \bar{P}$ denote the coupled process where $R(t)$ tracks the discrepancy or "second-class" particle. The joint generator is

$$\begin{aligned}(\bar{L}f)(\xi,r) = &\sum_j\sum_{i,i+j\neq r} p(j)\xi_i(1-\xi_{i+j})(f(\xi^{i,i+j},r)-f(\xi,r))\\ &+\sum_i p(-i)\xi_{r+i}(f(\xi^{r+i,r},r+i)-f(\xi,r))\\ &+\sum_i p(i)(1-\xi_{r+i})(f(\xi^{r,r+i},r+i)-f(\xi,r)).\end{aligned}$$

The first sum refers to jumps not involving the discrepancy location, while the second and third sums correspond to jumps of other particles to the discrepancy position and jumps of the discrepancy itself.

We have then

$$\begin{aligned}&E_\rho[(\eta_0(s)-\rho)(\eta_0(0)-\rho)]\\ &\quad= \rho(1-\rho)[P_\rho(\eta_0(s)=1|\eta_0(0)=1) - P_\rho(\eta_0(s)=1|\eta_0(0)=0)]\\ &\quad= \rho(1-\rho)\bar{P}[R(s)=0] \qquad (1.2)\end{aligned}$$

which leads to the relation mentioned in the prolog between occupation time variance and expected occupation at the origin of a second-class particle:

$$\lim_{t\to\infty} \sigma_t^2/t = \lim_{t\to\infty} 2\rho(1-\rho)\int_0^t (1-s/t)\bar{P}[R(s)=0]ds$$
$$= 2\rho(1-\rho)\int_0^\infty \bar{P}[R(s)=0]ds; \qquad (1.3)$$

the notation $p_t(0, j)$ given earlier now reads $p_t(0, j) = \bar{P}(R(t) = j)$.

The second-class particle process $R(t)$, with respect to its own history, is not Markov except when the jump rate p is symmetric, in which case, it is a symmetric random walk. In general, it is highly dependent on the whole system. However, one can roughly think of $R(t)$ as some sort of random walk with mean drift $(1-2\rho)t\sum ip(i)$. This drift vanishes exactly when p is either mean-zero ($\sum ip(i) = 0$) or $\rho = 1/2$, and so one might think the process is recurrent exactly in this case so that

$$\lim_{t\to\infty} \sigma_t^2/t = \infty \ \text{ in } d \le 2 \text{ when } p \text{ mean-zero} \quad \text{or } \rho = 1/2$$
$$< \infty \ \text{ otherwise.}$$

This has been established in all cases but one (cf. Refs. (1, 3, 8, 9)). What remains is to show the variance is superdiffusive in $d = 2$ when $\rho = 1/2$ and $\sum ip(i) \neq 0$ which is the point of this note.

Also, of key interest is how fast σ_t^2/t diverges in $d \le 2$ when p mean-zero or $\rho = 1/2$. In fact, it has been shown that $\sigma_t^2 \sim t^{3/2}$ and $t \log t$ in $d = 1$ and $d = 2$ respectively when p is mean-zero[3,9]. When p has a drift ($\sum ip(i) \neq 0$) and $\rho = 1/2$, as mentioned at the beginning of the article, σ_t^2 is conjectured to diverge as $t^{4/3}$ and $t(\log t)^{2/3}$ in $d = 1$ and $d = 2$ respectively. Indeed, as also mentioned, a lower bound on order $t^{5/4}$ has been shown in Ref. (1) in $d = 1$.

The main result of this note (Proposition 1.3) is to compute in $d = 2$ when p has a non-zero drift and $\rho = 1/2$ that

$$\liminf_{\lambda\to 0} \frac{\lambda^2}{\log(|\log\lambda|)}\int_0^\infty e^{-\lambda t}\sigma_t^2 dt > 0.$$

When the drift $\sum ip(i)$ is in the direction of one of the axes, the same result holds with "$\log|\log\lambda|$" replaced by "$|\log\lambda|^{1/2}$." Clearly σ_t^2/t diverges regardless, and moreover a formal Tauberian analogy would suggest that σ_t^2 is at least on order $t\log(\log t)$ in the general case and $t(\log t)^{1/2}$ in the more special case.

We mention that some general rough upper bounds in $d = 1, 2$, which cover the case when $\rho = 1/2$ and p has a drift, are easy to obtain by a comparison with the symmetrized process, namely $\sigma_t^2 \le c_1 t^{3/2}$ in $d = 1$ and $\sigma_t^2 \le c_2 t\log t$

in $d = 2$. Although well known, we include them for completeness in Proposition 1.2.

1.3. Variational Formulas

The method of proof does not work with second-class particles, but with certain variational formulas for some resolvent quantities. The generator L, with respect to P_ρ, can be decomposed into symmetric and anti-symmetric parts, $L = S + A$ where $S = (L + L^*)/2$ and $A = (L - L^*)/2$. One can check that the symmetric operator S is in fact the generator of simple exclusion with symmetrized jump probabilities $(p(\cdot) + p(-\cdot))/2$.

Consider now the resolvent operators $(\lambda - L)^{-1} : L^2(P_\rho) \to L^2(P_\rho)$ and $(\lambda - S)^{-1} : L^2(P_\rho) \to L^2(P_\rho)$ well defined for $\lambda > 0$. In particular, $(\lambda - L)^{-1} f = \int_0^\infty e^{-\lambda s}(T_s f)ds$ and $(\lambda - S)^{-1} f = \int_0^\infty e^{-\lambda s}(T_s^S f)ds$ where T_t^S is the semigroup for the process generated by S. Define, for local ϕ, the $H_{1,\lambda,L}$ norm $\|\cdot\|_{1,\lambda,L}$ by

$$\|\phi\|_{1,\lambda,L}^2 = \langle \phi, (\lambda - S)\phi\rangle_\rho + \langle A\phi, (\lambda - S)^{-1} A\phi\rangle_\rho$$

where we note terms $\langle \phi, (-S)\phi\rangle_\rho, \langle A\phi, (\lambda - S)^{-1} A\phi\rangle_\rho \geq 0$ as $-S$ is a non-negative operator. The $H_{1,\lambda,L}$ Hilbert space is then the completion over local functions with respect to this norm.

To define a dual norm, consider for $f \in L^2(P_\rho)$ and local ϕ that $\langle f, \phi\rangle_\rho \leq \|f\|_0\|\phi\|_0 \leq \lambda^{-1/2}\|f\|_0\|\phi\|_{1,\lambda,L}$. Then, the dual norm $\|f\|_{1,\lambda,L}$ given by

$$\|f\|_{-1,\lambda,L} = \sup_{\substack{\phi \text{ local} \\ \|\phi\|_{1,\lambda,L}=1}} \langle f, \phi\rangle_\rho \tag{1.4}$$

is always finite with bound $\|f\|_{-1,\lambda,L}^2 \leq \lambda^{-1}\|f\|_0^2$. Let $H_{-1,\lambda,L}$ be the corresponding Hilbert space with respect to $\|\cdot\|_{-1,\lambda,L}$. An equivalent variational form for $\|f\|_{-1,\lambda,L}$, which will be useful, is given as follows.

$$\|f\|_{-1,\lambda,L}^2 = \sup_{\phi \text{ local}} \left\{2\langle f, \phi\rangle_\rho - \langle \phi, (\lambda - S)\phi\rangle_\rho - \langle A\phi, (\lambda - S)^{-1} A\phi\rangle_\rho\right\}. \tag{1.5}$$

We now evaluate these variational expressions in closed form; a proof of the following lemma is given in Subsection 2.5 to be complete. See also Section 4.5 Ref. (6) in this context.

Lemma 1.1. *For $f \in L^2(P_\rho)$ and $\lambda > 0$, we have*

$$\|f\|_{-1,\lambda,L}^2 = \langle f, (\lambda - L)^{-1} f\rangle_\rho. \tag{1.6}$$

It will be helpful also to define the $H_{1,\lambda}$ norm $\|g\|_{1,\lambda}$ for local g by $\|g\|_{1,\lambda}^2 = \langle g, (\lambda - S)g\rangle_\rho$. For $f \in L^2(P_\rho)$ define also the dual $H_{-1,\lambda}$ norm $\|f\|_{-1,\lambda}$ given by

$\|f\|^2_{-1,\lambda} = \langle f, (\lambda - S)^{-1} f\rangle_\rho$. Then, in this notation, (1.5), noting (1.6), is rewritten as

$$\langle f, (\lambda - L)^{-1} f\rangle_\rho = \sup_{g \text{ local}} \left\{2\langle f, g\rangle_\rho - \|g\|^2_{1,\lambda} - \|Ag\|^2_{-1,\lambda}\right\}. \tag{1.7}$$

Also $\langle f, (\lambda - S)^{-1} f\rangle_\rho = \sup_{g \text{ local}}\{2\langle f, g\rangle_\rho - \|g\|^2_{1,\lambda}\}$.

1.4. Connection Between $\langle \eta_0 - \rho, (\lambda - L)^{-1}(\eta_0 - \rho)\rangle_\rho$ and σ_t^2

At this point, we note an explicit relation between $\|\eta_0 - \rho\|^2_{-1,\lambda,L}$ and σ_t^2. Compute, observing (1.1), that

$$\begin{aligned}\langle \eta_0 - \rho, (\lambda - L)^{-1}(\eta_0 - \rho)\rangle_\rho &= \int_0^\infty e^{-\lambda t}\langle \eta_0 - \rho, T_t(\eta_0 - \rho)\rangle_\rho \, dt \\ &= \int_0^\infty e^{-\lambda t} E_\rho[(\eta_0(0) - \rho)(\eta_0(t) - \rho)]\, dt \\ &= \frac{\lambda^2}{2}\int_0^\infty e^{-\lambda t}\sigma_t^2\, dt.\end{aligned}$$

1.5. Upper Bounds

Well known upperbounds on σ_t^2 follow from two statements which we include here for completeness.

Proposition 1.1. *There is a universal constant C_1 such that*

$$\begin{aligned}\sigma_t^2 &\leq C_1 t\langle \eta_0 - \rho, (t^{-1} - L)^{-1}(\eta_0 - \rho)\rangle_\rho \\ &\leq C_1 t\langle \eta_0 - \rho, (t^{-1} - S)^{-1}(\eta_0 - \rho)\rangle_\rho.\end{aligned}$$

Proof: The first line is well-known (cf. Lemma 3.9 Ref. (9)), and the second follows by dropping the non-negative term "$\langle A\phi, (\lambda - S)^{-1} A\phi\rangle_\rho$" from (1.7).

The next proposition is proved in Ref. (3).

Proposition 1.2. *In $d \leq 2$, there exists a constant $C_2 = C_2(d, \rho, p)$ where for large t,*

$$\langle \eta_0 - \rho, (t^{-1} - S)^{-1}(\eta_0 - \rho)\rangle_\rho \;\leq\; \begin{cases} C_2\sqrt{t} & \text{in } d = 1 \\ C_2 \log t & \text{in } d = 2\end{cases}$$

and so by Proposition 1.1, $\sigma_t^2 \leq C_1 C_2 t^{3/2}$ in $d = 1$ and $C_1 C_2 t \log t$ in $d = 2$.

1.6. Lower Bounds

The lowerbounds are through variational formulas and the "connection" between $\|\eta_0 - \rho\|_{-1,\lambda,L}$ and σ_t^2 remarked in Subsection 1.4. The following is the main result of this note and is proved in Subsection 2.6. Let e_1 and e_2 denote the standard basis in $\mathbb{R}^2$.

Proposition 1.3. *In $d = 2$, when $\sum ip(i) \neq 0$ and $\rho = 1/2$, there is a constant $C_3 = C_3(p)$ where for all small $\lambda > 0$,*

$$\frac{\lambda^2}{2}\int_0^\infty e^{-\lambda t}\sigma_t^2 dt = \langle \eta_0 - 1/2, (\lambda - L)^{-1}(\eta_0 - 1/2)\rangle_{\frac{1}{2}} \geq C_3 \log(|\log \lambda)|);$$

when, more specifically, $\sum ip(i) = ce_1$ or ce_2 is a non-zero multiple of either e_1 or e_2, $\langle \eta_0 - 1/2, (\lambda - L)^{-1}(\eta_0 - 1/2)\rangle_{\frac{1}{2}} \geq C_3|\log \lambda|^{1/2}$.

2. SOME PRELIMINARIES

We first give some tools and definitions before going to the proof of Proposition 1.3 in Subsection 2.6. In the following, the dimension $d = 2$ is fixed.

2.1. Comparison Bound

We compare $\langle f, (\lambda - L)^{-1} f\rangle_\rho$ with the formula with respect to a "nearest-neighbor" operator L_0. Let $m_i = e_i \cdot \sum jp(j)$ for $i = 1, 2$. As the drift of p is assumed not to vanish, at least one of the m_i's is not zero. Without loss of generality, suppose $m_1 \neq 0$.

Let L_0 be the exclusion generator corresponding to nearest-neighbor jump rates $p_0(\cdot)$ where

$$p_0(e_1) = |m_1|, p_0(e_2) = |m_2|, \quad \text{and } p_0(i) = 0 \text{ otherwise}, \quad \text{when } m_2 \neq 0 \quad \text{and}$$
$$p_0(e_1) = |m_1|, p_0(\pm e_2) = 1/4, \quad \text{and } p_0(i) = 0 \text{ otherwise}, \quad \text{when } m_2 = 0.$$

The following is proved in Theorem 2.1 Ref. (10).

Proposition 2.1. *There is a constant $C_4 = C_4(p)$ where*

$$C_4^{-1}\langle f, (\lambda - L_0)^{-1} f\rangle_\rho \leq \langle f, (\lambda - L)^{-1} f\rangle_\rho \leq C_4\langle f, (\lambda - L_0)^{-1} f\rangle_\rho.$$

2.2. Duality

Let $\mathcal{E}$ denote the collection of finite subsets of $\mathbb{Z}^2$, and let $\mathcal{E}_n$ denote those subsets of cardinality n. Let also Ψ_B be the function

$$\Psi_B(\eta) \;=\; \prod_{x\in B} \frac{\eta_x - \rho}{\sqrt{\rho(1-\rho)}}$$

where we take $\Psi_\emptyset = 1$ by convention. One can check that $\{\Psi_B : B \in \mathcal{E}\}$ is Hilbert basis of $L^2(P_\rho)$. In particular, any function $f \in L^2(P_\rho)$ has decomposition

$$f = \sum_{n\geq 0} \sum_{B\in\mathcal{E}_n} \mathfrak{f}(B)\Psi_B$$

with coefficient $\mathfrak{f} : \mathcal{E} \to \mathbb{R}$ which in general depends on ρ. Then, for $f, g \in L^2(P_\rho)$,

$$\langle \mathfrak{f}, \mathfrak{g}\rangle \;:=\; \langle f, g\rangle_\rho = \sum_{B\in\mathcal{E}} \mathfrak{f}(B)\mathfrak{g}(B)$$

and $\|\mathfrak{f}\|^2 := \|f\|_0^2 = \langle f, f\rangle_\rho$. Let also $\mathcal{C}_n$ be the subspace generated by finite linear combinations of $\{\Psi_B : |B| = n\}$. When $f \in \mathcal{C}_n$, we have $\mathfrak{f}$ is a function on $\mathcal{E}_n$, and we say in this case both f and its coefficient $\mathfrak{f}$ are of degree n. Note also, when f is local, then $\mathfrak{f}$ is also local on $\mathcal{E}$, that is with support on a finite number of subsets of $\mathbb{Z}^2$.

The operators L, S and A have counterparts $\mathfrak{L}$, $\mathfrak{S}$ and $\mathfrak{A}$ which act on "coefficient" functions $\mathfrak{f}$. These are given in the expressions

$$Lf = \sum_{B\in\mathcal{E}} (\mathfrak{L}\mathfrak{f})(B)\Psi_B, \quad Sf = \sum_{B\in\mathcal{E}} (\mathfrak{S}\mathfrak{f})(B)\Psi_B, \quad \text{and} \quad Af \;=\; \sum_{B\in\mathcal{E}} (\mathfrak{A}\mathfrak{f})(B)\Psi_B.$$

Let s and a be the symmetric and anti-symmetric parts of p, $s(i) = (p(i) + p(-i))/2$ and $a(i) = (p(i) - p(-i))/2$. Also for $B \subset \mathbb{Z}^d$, denote

$$B_{x,y} = \begin{cases} B\setminus\{x\}\cup\{y\} & \text{when } x \in B, y \notin B \\ B\setminus\{y\}\cup\{x\} & \text{when } x \notin B, y \in B \\ B & \text{otherwise.} \end{cases}$$

Now, of course, $\mathfrak{L} = \mathfrak{S} + \mathfrak{A}$. Moreover, the symmetric part $\mathfrak{S}$ can be computed as

$$(\mathfrak{S}\mathfrak{f})(B) \;=\; \frac{1}{2} \sum_{x,y\in\mathbb{Z}^2} s(y-x)[\mathfrak{f}(B_{x,y}) - \mathfrak{f}(B)].$$

Note that $\mathfrak{S}\mathfrak{f} \in \mathcal{C}_n$ for $\mathfrak{f} \in \mathcal{C}_n$, and so $\mathfrak{S}$ "preserves" degrees. Moreover, $\mathfrak{S}$ restricted to degree n functions governs the dynamics of the set of coordinates of n particles in symmetric simple exclusion.

Also, the anti-symmetric part $\mathfrak{A}$, after a now standard careful calculation (cf. Ref. (11)), can be decomposed into the sum of three operators which preserve, increase, and decrease the degree of the function acted upon: $\mathfrak{A} = (1-2\rho)\mathfrak{A}_0 + 2\sqrt{\rho(1-\rho)}(\mathfrak{A}^+ - \mathfrak{A}^-)$.

$$(\mathfrak{A}_0\mathfrak{f})(B) = \sum_{\substack{x\in B\\ y\notin B}} a(y-x)[\mathfrak{f}(B_{x,y}) - \mathfrak{f}(B)]$$
$$(\mathfrak{A}^+\mathfrak{f})(B) = \sum_{\substack{x\in B\\ y\in B}} a(y-x)\mathfrak{f}(B - \{y\})$$
$$(\mathfrak{A}^-\mathfrak{f})(B) = \sum_{\substack{x\notin B\\ y\notin B}} a(y-x)\mathfrak{f}(B \cup \{x\}).$$

As $\mathfrak{A}_0$, $\mathfrak{A}^+$ and $\mathfrak{A}^-$ take a degree n coefficient function $\mathfrak{f} : \mathcal{E}_n \to \mathbb{R}$ into respectively a degree n, $n+1$ and $n-1$ function. It will be helpful to write $\mathfrak{A}$ in terms of its "degree" actions,

$$\mathfrak{A} \;=\; \sum_{n\geq 0}(\mathfrak{A}_{n,n-1} + \mathfrak{A}_{n,n} + \mathfrak{A}_{n,n+1})$$

where $\mathfrak{A}_{m,n}$ is the part which takes degree m functions to degree n functions. Here, by convention $\mathfrak{A}_{0,-1} = \mathfrak{A}_{0,0} = \mathfrak{A}_{1,0} = 0$ are zero operators.

At this point, when $\rho = 1/2$, we observe that $\mathfrak{A} = \mathfrak{A}^+ - \mathfrak{A}^-$ as the part which preserves degree vanishes here.

2.3. H_1 and H_{-1} Coefficent Norms

Define, for local functions $\mathfrak{g}$, the $H_{1,\lambda}$ norm $\|\mathfrak{g}\|_{1,\lambda}$ by $\|\mathfrak{g}\|_{1,\lambda}^2 = \langle \mathfrak{g}, (\lambda - \mathfrak{S})\mathfrak{g}\rangle$, and also the dual $H_{-1,\lambda}$ norm $\|\mathfrak{g}\|_{-1,\lambda}$ by $\|\mathfrak{g}\|_{-1,\lambda}^2 = \sup_{\mathfrak{h}\text{ local}}\{2\langle \mathfrak{g}, \mathfrak{h}\rangle - \|\mathfrak{h}\|_{1,\lambda}^2\}$. Then, with respect to coefficient functions $\mathfrak{g}$, we have (1.7) is rewritten as

$$\langle f, (\lambda - L)^{-1} f\rangle_\rho = \sup_{\mathfrak{g}\text{ local}} \left\{ 2\langle \mathfrak{f}, \mathfrak{g}\rangle - \|\mathfrak{g}\|_{1,\lambda}^2 - \|\mathfrak{A}\mathfrak{g}\|_{-1,\lambda}^2 \right\}. \tag{2.8}$$

2.4. "Free Particle" Bounds

To analyze these variational formulas, it will be helpful computationally to "remove the hard-core exclusion." In other words, we want to get equivalent bounds in terms of operators which govern completely independent or "free" motions. We follow Bernardin[1]. Let $\chi_n = (\mathbb{Z}^2)^n$ and consider n independent random walks with jump rates s on $\mathbb{Z}^2$. The process $x_t = (x_t^1, \ldots, x_t^n)$ evolves on χ_n and has generator acting on local, namely, finitely supported functions,

$$(\mathfrak{S}_{\text{free}}\mathfrak{f})(x) \;=\; \sum_{\substack{1\leq j\leq n\\ z\in\mathbb{Z}^2}} s(z)[\mathfrak{f}(x + z\omega_j) - \mathfrak{f}(x)]$$

where $z\omega_j = (0, \ldots, 0, z, 0, \ldots, 0)$ is the state with z in the jth place. With respect to local functions on χ_n, define innerproduct

$$\langle \phi, \psi\rangle_{\text{free}} = \frac{1}{n!}\sum_{x\in\chi_n} \phi(x)\psi(x),$$

and define $H_{1,\lambda}$ and $H_{-1,\lambda}$ norms $\|\phi\|_{1,\lambda,\mathrm{free}} = \langle \phi, (\lambda - \mathfrak{S}_{\mathrm{free}})\phi\rangle_{\mathrm{free}}^{1/2}$ and $\|\phi\|_{-1,\lambda,\mathrm{free}}$ by

$$\|\phi\|^2_{-1,\lambda,\mathrm{free}} = \sup_{\psi \text{ local on } \chi_n} \left\{ 2\langle \phi, \psi\rangle_{\mathrm{free}} - \|\psi\|^2_{1,\lambda,\mathrm{free}} \right\}.$$

Let now $\mathcal{G}_n \subset \chi_n$ be those points whose coordinates are distinct. For a function $\mathfrak{f} \in \mathcal{C}_n$, consider its natural extension to a function $\mathfrak{f}_{\mathrm{free}}$ on χ_n, namely $\mathfrak{f}_{\mathrm{free}}(x) = \mathfrak{f}(U)$ where U is the set formed from the coordinates of $x \in \chi_n$. Note that $\mathfrak{f}_{\mathrm{free}}$ is supported on $\mathcal{G}_n$. The following is a part of Theorems 3.1 and 3.2[(1)] [which simplifies as $\tilde{\mathfrak{f}} = \mathfrak{f}_{\mathrm{free}}$ for $\mathfrak{f} \in \mathcal{C}_1$, and $1_{x\in\mathcal{G}_n}\tilde{\mathfrak{f}} = \mathfrak{f}_{\mathrm{free}}$ for $\mathfrak{f} \in \mathcal{C}_n$].

Proposition 2.2. *There exists a constant C_5, depending only on the function degree, such that for $\mathfrak{f} \in \mathcal{C}_1$ we have*

$$C_5^{-1}\|\mathfrak{f}_{\mathrm{free}}\|^2_{1,\lambda,\mathrm{free}} \le \|\mathfrak{f}\|^2_{1,\lambda} \le C_5\|\mathfrak{f}_{\mathrm{free}}\|^2_{1,\lambda,\mathrm{free}}.$$

Also, for $\mathfrak{f} \in \mathcal{C}_n$ (for any $n \ge 1$),

$$\|\mathfrak{f}\|^2_{-1,\lambda} \le C_5\|\mathfrak{f}_{\mathrm{free}}\|^2_{-1,\lambda,\mathrm{free}}.$$

We express now the "free" $H_{1,\lambda}$ and $H_{-1,\lambda}$ norms in terms of Fourier transforms. Let ψ be a local function on χ_n and let $\widehat{\psi}$ be its Fourier transform

$$\widehat{\psi}(s_1, \ldots, s_n) = \frac{1}{\sqrt{n!}} \sum_{x\in\chi_n} e^{2\pi i(x_1\cdot s_1 + \cdots + x_n\cdot s_n)}\psi(x)$$

where $s_1, \ldots, s_n \in ([0,1]^2$. Compute

$$\widehat{\mathfrak{S}_{\mathrm{free}}\psi}(s_1, \ldots, s_n) = -\left[\sum_{j=1}^n \theta_2(s_j)\right]\widehat{\psi}(s_1, \ldots, s_n)$$

where $\theta_2(u) = 2\sum_{z\in\mathbb{Z}^2} s(z)\sin^2(\pi(u\cdot z))$. Hence, we have

$$\|\psi\|^2_{1,\lambda,\mathrm{free}} = \int_{\substack{s\in([0,1]^2)^n\\ s=(s_1,\ldots,s_n)}} \left(\lambda + \sum_{j=1}^n \theta_2(s_j)\right)|\widehat{\psi}(s_1, \ldots, s_n)|^2 ds$$

and

$$\|\psi\|^2_{-1,\lambda,\mathrm{free}} = \int_{\substack{s\in([0,1]^2)^n\\ s=(s_1,\ldots,s_n)}} \frac{|\widehat{\psi}(s_1, \ldots, s_n)|^2}{\lambda + \sum_{j=1}^n \theta_2(s_j)} ds.$$

2.5. Proof of Lemma 1.1

The proof follows in two steps. In step 1, as $u = (\lambda - L)^{-1}f$ belongs to the domain of L, let $\{u_n\}$ be local functions such that $\lim u_n = u$ and

$\lim_n Lu_n = Lu$ in $L^2(P_\rho)$. Then, for ϕ local, write $\langle f, \phi\rangle_\rho = \langle(\lambda - L)u, \phi\rangle_\rho = \lim_n\langle(\lambda - L)u_n, \phi\rangle_\rho = \lim_n\langle u_n, (\lambda - L^*)\phi\rangle_\rho$ and

$$\langle u_n, (\lambda - L^*)\phi\rangle_\rho \le \langle u_n, (\lambda - S)u_n\rangle_\rho^{1/2}\langle(\lambda - L^*)\phi, (\lambda - S)^{-1}(\lambda - L^*)\phi\rangle_\rho^{1/2}.$$

As $\langle u_n, (\lambda - S)u_n\rangle_\rho = \langle u_n, (\lambda - L)u_n\rangle \to \langle f, (\lambda - L)^{-1} f\rangle_\rho$ and $\langle(\lambda - L^*)\phi, (\lambda - S)^{-1}(\lambda - L^*)\phi\rangle_\rho = \|\phi\|^2_{1,\lambda,L}$ we have $\langle f, \phi\rangle_\rho \le \langle f, (\lambda - L)^{-1} f\rangle_\rho^{1/2} \|\phi\|_{1,\lambda,L}$. Hence, $\|f\|^2_{-1,\lambda,L} \le \langle f, (\lambda - L)^{-1} f\rangle_\rho$.

In step 2, define $v_n = (\lambda - L^*)^{-1}(\lambda - S)u_n$ in the domain of L^*, and let $\{v_{m,n}\}$ be a sequence of local functions where $\lim_m v_{m,n} = v_n$ and $\lim_m L^*v_{m,n} = L^*v_n$ in $L^2(P_\rho)$. Then, $\lim_n \lim_m\langle f, v_{m,n}\rangle_\rho = \lim_n\langle(\lambda - L)^{-1} f, (\lambda - S)u_n\rangle_\rho = (1/2)\lim_n\langle u, (\lambda - L)u_n\rangle_\rho + (1/2)\lim_n\langle(\lambda - L)u, u_n\rangle_\rho = \langle f, (\lambda - L)^{-1} f\rangle_\rho$. Also, $\lim_n \lim_m \|v_{m,n}\|^2_{1,\lambda,L} = \lim_n \lim_m \langle(\lambda - L^*)v_{m,n}, (\lambda - S)^{-1} (\lambda - L^*)v_{m,n}\rangle_\rho = \lim_n \langle(\lambda - S)u_n, u_n\rangle_\rho = \lim_n \langle(\lambda - L)u_n, u_n\rangle_\rho = \langle f, (\lambda - L)^{-1} f\rangle_\rho$. Hence, substituting into (1.4), we get $\|f\|^2_{-1,\lambda,L} \ge \langle f, (\lambda - L)^{-1} f\rangle_\rho$. □

2.6. Proof of Proposition 1.3

Let $\rho = 1/2$ and $f(\eta) = \eta_0 - 1/2$. To prove Proposition 1.1, we find lower bounds on $\|f\|^2_{-1,\lambda} = \langle f, (\lambda - L)^{-1} f\rangle_{1/2}$. From Proposition 2.1, we will assume without loss of generality that L takes nearest-neighbor form $L = L_0$. Consider now the variational formula given in (2.8). The strategy will be (1) to replace to restrict the supremum there to local degree 1 functions, and (2) to use the comparison bounds with respect to independent particles (Proposition 2.2) to help bound terms in the formula.

To this end, note $\mathfrak{f} = (1/2)1_{\{0\}}$ where $1_{\{0\}}$ is the indicator of the singleton $\{0\}$. Let $\phi \in \mathcal{C}_1$ be a local function on $\mathcal{E}_1$, and observe then ϕ_{free} is local on $\mathbb{Z}^2$. To simplify notation, let $s(\pm e_1) = b_1 > 0, s(\pm e_2) = b_2 > 0$ and $a(e_1) = -a(-e_1) = a_1 \neq 0, a(e_2) = -a(-e_2) = a_2$. As $\rho = 1/2$, $\mathfrak{A}\phi$ takes simple form $\mathfrak{A}\phi = \mathfrak{A}_{1,2}\phi$. More specifically, $(\mathfrak{A}_{1,2}\phi)(\{x, y\}) = a(y - x)(\phi(\{x\}) - \phi(\{y\}))$, and $(\mathfrak{A}_{1,2}\phi)_{\text{free}}$ is written as

$$(\mathfrak{A}_{1,2}\phi)_{\text{free}}((x, y)) = \begin{cases} \pm a_1(\phi_{\text{free}}(x) - \phi_{\text{free}}(x \pm e_1)) & \text{when } y = x \pm e_1 \\ \pm a_2(\phi_{\text{free}}(x) - \phi_{\text{free}}(x \pm e_2)) & \text{when } y = x \pm e_2 \\ 0 & \text{otherwise.} \end{cases}$$

Inserting into (2.8), using Proposition 2.2, we have for some constant C_6 that

$$\langle f, (\lambda - L)^{-1} f\rangle_{1/2} \ge C_6 \sup_{\phi\in\mathcal{C}_1,\ \text{local}} \times \left\{\langle 1_{\{0\}}, \phi\rangle - \|\phi_{\text{free}}\|^2_{1,\lambda,\text{free}} - \|(\mathfrak{A}_{1,2}\phi)_{\text{free}}\|^2_{-1,\lambda,\text{free}}\right\}. \qquad (2.9)$$

Now, it is a calculation to find for $s = (s_1, s_2), t = (t_1, t_2) \in [0, 1]^2$ that

$$
\begin{aligned}
(\widehat{\mathfrak{A}_{1,2}\phi})_{\text{free}}(s, t) &= \frac{1}{\sqrt{2}} \sum_{(x,y)\in(\mathbb{Z}^2)^2} e^{2\pi i(x\cdot s+y\cdot t)}(\mathfrak{A}_{1,2}\phi)_{\text{free}}((x, y)) \\
&= \frac{i}{\sqrt{2}}\widehat{\phi_{\text{free}}}(s + t)[2a_1 \sin(2\pi s_1) + 2a_2 \sin(2\pi s_2) \\
&\quad + 2a_1 \sin(2\pi t_1) + 2a_2 \sin(2\pi t_2)].
\end{aligned}
$$

Also, note $\langle 1_{\{0\}}, \phi\rangle = \phi_{\text{free}}(0)$. Then, the expression in brackets in (2.9) in Fourier terms is

$$
\begin{aligned}
&\int_{[0,1]^2} \left(\widehat{\phi_{\text{free}}}(s) - (\lambda + \theta_2(s))|\widehat{\phi_{\text{free}}}|^2(s)\right) ds \qquad (2.10) \\
&- \frac{1}{2}\int_{([0,1]^2)^2} \frac{\left[\sum_{i=1}^{2} 2a_i \sin(2\pi s_i) + 2a_i \sin(2\pi t_i)\right]^2}{\lambda + \theta_2(s) + \theta_2(t)} |\widehat{\phi_{\text{free}}}(s + t)|^2 ds_1 ds_2 dt_1 dt_2.
\end{aligned}
$$

In the following, to simplify notation, we will drop the subscript "free."

We now change coordinates in the second integral:

$$
(s_1, s_2, t_1, t_2) = \left(\frac{u + v}{2}, \frac{w + z}{2}, \frac{u - v}{2}, \frac{w - z}{2}\right)
$$

(whose Jacobian determinant in absolute value is 1/4). The region $[0, 1]^4$ is mapped to D^2 where D is a planar diamond with vertices $(0, 0), (1, -1), (1, 1), (2, 0)$. Let

$$
\begin{aligned}
\gamma(u, v, w, z) &= 4b_1 \sin^2\left(\pi\frac{u + v}{2}\right) + 4b_2 \sin^2\left(\pi\frac{w + z}{2}\right) \\
&\quad + 4b_1 \sin^2\left(\pi\frac{u - v}{2}\right) + 4b_2 \sin^2\left(\pi\frac{w - z}{2}\right) \\
&= 8b_1 \sin^2(\pi(u/2)) \cos^2(\pi(v/2)) + 8b_1 \sin^2(\pi(v/2)) \cos^2(\pi(u/2)) \\
&\quad + 8b_2 \sin^2(\pi(w/2)) \cos^2(\pi(z/2)) + 8b_2 \sin^2(\pi(z/2)) \cos^2(\pi(w/2))
\end{aligned}
$$

and

$$
\begin{aligned}
\upsilon(u, v, w, z) &= 16a_1^2 \sin^2(\pi u) \cos^2(\pi v) + 16a_2^2 \sin^2(\pi w) \cos^2(\pi z) \\
&\quad + 32a_1a_2 \sin(\pi u) \cos(\pi v) \sin(\pi w) \cos(\pi z) \\
&\leq 32a_1^2 \sin^2(\pi u) \cos^2(\pi v) + 32a_2^2 \sin^2(\pi w) \cos^2(\pi z).
\end{aligned}
$$

The integral in the second term of (2.10), as $\theta_2((u,v)) = 4b_1\sin^2(\pi u) + 4b_2\sin^2(\pi v)$, is rewritten as

$$\frac{1}{4}\int_D\int_D \frac{\upsilon(u,v,w,z)}{\lambda+\gamma(u,v,w,z)}|\widehat{\phi}((u,w))|^2 du\,dv\,dw\,dz$$
$$\leq \frac{32}{4}\int_D\int_D \frac{a_1^2\sin^2(\pi u)\cos^2(\pi v)+a_2^2\sin^2(\pi w)\cos^2(\pi z)}{\lambda+\gamma(u,v,w,z)}$$
$$\times|\widehat{\phi}((u,w))|^2 du\,dv\,dw\,dz.$$

By changing variables and using some symmetries–for instance the marginal $du\,dv$-integration over $1\le u\le 2, 0\le \pm v\le 2-u$, and $0\le u\le 1, -u\le v\le 0$, noting $\widehat{\phi}((u+1,w))) = \widehat{\phi}((u,w))$, is the same as over $0\le u\le 1, u\le v\le 1$, and $0\le u\le 1$, $0\le v\le u$ respectively–it is not difficult to see that the last integral equals

$$32\int_{[0,1]^2}\int_{[0,1]^2}\frac{a_1^2\sin^2(\pi u)\cos^2(\pi v)+a_2^2\sin^2(\pi w)\cos^2(\pi z)}{\lambda+\gamma(u,v,w,z)}$$
$$\times|\widehat{\phi}((u,w))|^2 dv\,dz\,du\,dw$$
$$= 32\int_0^1\int_0^1 [a_1^2\sin^2(\pi u)F_\lambda^1(u,w)+a_2^2\sin^2(\pi w)F_\lambda^2(u,w)]|\widehat{\phi}((u,w))|^2 du dw$$

where

$$F_\lambda^1(u,w) = \int_0^1\int_0^1 \frac{\cos^2(\pi v)dv\,dz}{\lambda+\gamma(u,v,w,z)} \text{ and } F_\lambda^2(u,w) = \int_0^1\int_0^1 \frac{\cos^2(\pi z)dv\,dz}{\lambda+\gamma(u,v,w,z)}.$$

Let now $C_7 = 4(b_1+b_2)+16(a_1^2+a_2^2)$. Substituting into (2.10) and (2.9), we obtain $C_6^{-1}\langle f,(\lambda-L)^{-1}f\rangle_\rho$ greater than

$$\sup_\phi\Big\{\int_0^1\int_0^1 \widehat{\phi}((u,w)) \tag{2.11}$$
$$-\big[\lambda+C_7(\sin^2(\pi u)+\sin^2(\pi w))(1+F_\lambda^1(u,w)+F_\lambda^2(u,w))\big]|\widehat{\phi}((u,w))|^2 du\,dw\Big\}$$

where the supremum is on ϕ local, or without loss of generality on $L^2(\mathbb{Z}^2)$. When specifically $a_2 = 0$, we have the lower bound

$$\sup_\phi\Big\{\int_0^1\int_0^1 \widehat{\phi}((u,w)) \tag{2.12}$$
$$-\big[\lambda+C_7(\sin^2(\pi u)+\sin^2(\pi w))+C_7\sin^2(\pi u)F_\lambda^1(u,w)\big]|\widehat{\phi}((u,w))|^2 du\,dw\Big\}.$$

We now concentrate on the general rates case bound (2.11). By optimizing on the quadratic expression involving ϕ we get the lower bound

$$\frac{1}{4}\int_0^1\int_0^1 \frac{du\,dw}{\lambda + C_7(\sin^2(\pi u)+\sin^2(\pi w))(1+F^1_\lambda(u,w)+F^2_\lambda(u,w))} \tag{2.13}$$

with optimizer

$$\widehat{\phi}((u,w)) \;=\; \frac{1}{2\big[\lambda + C_7(\sin^2(\pi u)+\sin^2(\pi w))(1+F^1_\lambda(u,w)+F^2_\lambda(u,w))\big]}$$

which is the transform of a real function as $\widehat{\phi}((u,w)) = \widehat{\phi}^*((1-u,1-w))$ (note $F^i_\lambda(u,w) = F^i_\lambda(1-u,1-w)$ for $i=1,2$ by observing the form of γ, and changing variables $v\to 1-v$ and $z\to 1-z$).

We now bound $F^1_\lambda(u,w)+F^2_\lambda(u,w)$ for $|u|,|w|\le 1/2$ and $\lambda\le 1$. Since $\cos(x)$ is decreasing for $0\le x\le \pi/2$ and $\sin(x)\ge (2/\pi)x$ for $0\le x\le\pi/2$, we have

$$\begin{aligned}
F^1_\lambda(u,w)+F^2_\lambda(u,w) &\le \int_{[([1/2,1]\times[0,1])\cup([0,1]\times[1/2,1])} \frac{2dv\,dz}{\lambda+4b_1v^2+4b_2z^2}\\
&\qquad + \int_0^{1/2}\int_0^{1/2}\frac{2dv\,dz}{\lambda+4b_1(u^2+v^2)+4b_2(w^2+z^2)}\\
&\le C_8 + \int_0^{\pi/2}\int_0^1 \frac{2sdsd\alpha}{\lambda+\bar{b}(u^2+w^2+s^2)}\\
&= C_8 + \frac{\pi}{2\bar{b}}\log\Big[\frac{\lambda+\bar{b}(u^2+w^2)+\bar{b}}{\lambda+\bar{b}(u^2+w^2)}\Big]\\
&\le C_9 + \frac{\pi}{2\bar{b}}|\log(\lambda+\bar{b}(u^2+w^2))|.
\end{aligned}$$

where $\bar{b} = 4\min\{b_1,b_2\}$ and $C_8 = C_8(b_1,b_2)$, $C_9 = C_9(b_1,b_2)$ are constants.

Hence, as $\sin(x)\le x$, we can bound (2.13) below by

$$\frac{1}{4}\int_0^{1/2}\int_0^{1/2}\frac{du\,dw}{\lambda + C_7\pi^2(u^2+w^2)(1+C_9+(\pi/2\bar{b})|\log(\lambda+\bar{b}(u^2+w^2))|)}. \tag{2.14}$$

We have, with respect to constants C_{10}, C_{11}, for all small $\lambda>0$ that 4 times (2.14) is greater than

$$\begin{aligned}
&\int_0^{1/4}\frac{rdr}{\lambda + C_{10}r^2(1+|\log(\lambda+\bar{b}r^2)|)}\\
&\quad = \int_0^{1/(4\sqrt{\lambda})}\frac{rdr}{1+C_{10}r^2(1+|\log\lambda(1+\bar{b}r^2)|)}
\end{aligned}$$

$$\geq \int_1^{1/(4\sqrt{\lambda})} \frac{r\,dr}{r^2(1 + C_{10} + C_{10}|\log((1+\bar{b}r^2)/r^2)| + C_{10}|\log(\lambda r^2)|)}$$

$$\geq \frac{1}{C_{11}} \int_1^{1/(4\sqrt{\lambda})} \frac{dr}{r|\log \lambda r^2|} \geq \frac{1}{C_{11}} \int_{\sqrt{\lambda}}^{1/4} \frac{dr}{r|\log r^2|}.$$

This last expression is order $|\log(\log \lambda)|$. We conjecture, to get the larger expected order of $|\log \lambda|^{2/3}$, it seems one would need to optimize also over higher degree functions in (2.9). In such optimizations one would need to handle $\tilde{\phi}$ for degrees ≥ 2 which does not seem trivial.

We note in the specific case $a_2 = 0$, we bound (2.12) by

$$\frac{1}{4} \int_0^{1/2}\int_0^{1/2} \frac{du\,dw}{\lambda + C_7\pi^2(u^2+w^2) + C_7\pi^2u^2(C_9 + (\pi/2\bar{b})|\log(\lambda + \bar{b}(u^2+w^2))|)}.$$

Following closely the sequence to bound the second-class particle variance in $d = 2$ (cf. p. 470 Ref. (4)), we observe $|\log(\lambda + u^2 + w^2)| \leq |\log(\lambda + w^2)|$ for λ small and $0 \leq u, w \leq 1/2$. And so, we obtain a lower bound on order

$$\int_0^{1/2}\int_0^{1/2} \frac{du\,dw}{\lambda + u^2 + w^2 + u^2|\log(\lambda + w^2)|}.$$

With substitution $u = y(1 + |\log(\lambda + w^2)|)^{-1/2}$ the above expression is bounded below by

$$\int_0^{1/2}\int_0^{1/2} \frac{dy\,dw}{\lambda + y^2 + w^2}(1 + |\log(\lambda + w^2)|)^{-1/2}.$$

Changing to polar coordinates and restricting $\pi/6 \leq \alpha \leq \pi/4$, we get a lower bound on order as in Ref. (4)

$$\int_0^{1/20} \frac{r\,dr}{\lambda + r^2}|\log(\lambda + r^2)|^{-1/2} \geq C_{12}|\log \lambda|^{1/2}$$

for a constant C_{12}. □

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